# FORMAL FROBENIUS MANIFOLD STRUCTURE ON EQUIVARIANT COHOMOLOGY

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Abstract. For a closed Kähler manifold with a Hamiltonian action of a connected compact Lie group by holomorphic isometries, we construct a formal Frobenius manifold structure on the equivariant cohomology by exploiting a natural DGBV algebra structure on the Cartan model.

The notion of Frobenius manifolds was introduced by Dubrovin [11, 12]. It gives a coordinate free formulation of solutions to the WDVV equations. As surveyed in Manin [23], there are three major methods to construct solutions to WDVV equations. The first method involves the theory of quantum cohomology via Gromov-Witten invariants (or topological sigma model in physics literature), see e.g. Ruan-Tian [25] and Kontsevich-Manin [19]. The second method is Saito's theory of singularities (or Landau-Ginzburg model in physics literature). The third method exploits the so-called DGBV algebras, named after Gerstenhaber, Batalin and Vilkovisky. This last method first appeared in Barannikov-Kontsevich [2] in the context of extended moduli spaces of Calabi-Yau manifolds, based on the Kodaira-Spencer theory of gravity of Bershadsky-Cecotti-Ooguri-Vafa [5] which extends earlier works of Tian [26] and Todorov [27]. A detailed account of the construction for general DGBV algebras can be found in Manin [23].

GBV algebras have appeared in many places in Mathematics and Mathematical Physics, e.g. algebraic deformation theory and Hochschild cohomology (Gerstenhaber [13]), string theory (Lian-Zuckerman [20]), gauge theory (Batalin-Vilkovisky [3]), etc. However, examples of DGBV algebras in differential geometry were relatively rare. Earlier examples include Tian's formula [26] in deformation theory of Calabi-Yau manifolds and Koszul's operator in Poisson geometry [18]. But the recognizations of DGBV algebra structures in these theories seem to come later in e.g. Ran [24] and Xu [28] respectively. In a series of papers [6, 7, 8], the authors constructed many DGBV algebras from Kähler and hyperkähler manifolds, and showed that they satisfy the conditions to carry out the construction of formal Frobenius manifold structures on the cohomology. Also, it was shown that different DGBV algebra structures can yield the same solution to the WDVV equations. In particular, we get formal Frobenius manifold structures on the de Rham and Dolbeault cohomology of a closed Kähler manifold. In this paper, we carry over the same ideas to equivariant cohomology, in the case of closed Kähler manifolds with Hamiltonian actions of a Lie group by holomorphic isometries.

The main result in this paper is related to the equivariant quantum cohomology. There are three models to define equivariant cohomology: the Borel model, the Cartan model and the Weil model. In Givental-Kim [15], a version of quantum cohomology based on Borel model was suggested and the rigorous formulation appeared in Lu [21]. Some discussions of WDVV equations and Frobenius manifold

structure in equivariant quantum cohomology can be found in Givental [14]. Our construction of a formal Frobenius manifold structure on equivariant cohomology uses the Cartan model, which enables us to manipulate everything by differential forms. Differential geometers are familiar with the idea that choosing nice representatives of cohomology classes by differential forms may lead to more information. This idea proves useful again in our theory: as shown in §6, we can work over the ring given by the equivariant cohomology of a point, while Givental [14] has to use its fractional field.

The rest of the paper is arranged as follows. We review some definitions and the construction of formal Frobenius manifolds from DGBV algebras in §1. The Cartan model for equivariant cohomology is reviewed in §2. In §3, we construct a DGBV algebra structure on the Cartan model when the group preserves a Poisson structure. We begin in §4 the discussion of Hamiltonian actions. The main results appear in the more technical sections §5 and §6, where we restrict our attention to the Kähler case.

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## 1. DGBV algebras and formal Frobenius manifolds

1.1. **Frobenius algebra.** Let $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra, $H$ a free $\mathbf{k}$-module of finite rank, endowed with a commutative associative multiplication

$$\wedge: \quad H \otimes_{\mathbf{k}} H \to H.$$

We call $(H, \wedge)$ a *Frobenius algebra* if there is a symmetric nondegenerate bilinear form $(\cdot,\cdot) : H \otimes H \to \mathbf{k}$ such that

$$(X \wedge Y, Z) = (X, Y \wedge Z) \tag{1}$$

for any $X, Y, Z \in H$. Such a bilinear form $(\cdot,\cdot)$ is called an *invariant inner product* on $H$.

Take a basis $\{e_a\}$ of $H$. Let $\eta_{ab} = (e_a, e_b)$ and $(\eta^{ab})$ be the inverse matrix of $(\eta_{ab})$. Also let $\phi_{ab}^c$ be the structure constants defined by

$$e_a \wedge e_b = \phi_{ab}^c e_c.$$

It is clear that the constants $\eta_{ab}$ and $\phi_{ab}^c$'s completely determine the structure of the Frobenius algebra. When $(H, \wedge)$ has an identity 1, $\eta_{ab}$ and $\phi_{ab}^c$'s can be encoded in a symmetric 3-tensor $\phi \in S^3H^*$ as follows. Assume that $e_0 = 1$. Set $\phi_{abc} = \phi_{ab}^p \eta_{pc}$. Then

$$\phi_{abc} = (e_a \wedge e_b, e_c).$$

From (1), one sees that $\phi$ is symmetric in the three indices. One can recover the inner product and the multiplication from $\phi$ since

$$\eta_{ab} = \phi_{0ab}, \quad \phi_{ab}^c = \phi_{abp}\eta^{pc}.$$

The associativity of the multiplication is equivalent to the following system of equations

$$\phi_{abp}\eta^{pq}\phi_{qcd} = \phi_{bcp}\eta^{pq}\phi_{aqd}. \tag{2}$$

1.2. **WDVV equations and Frobenius manifolds.** Let $(H, \wedge, (\cdot,\cdot))$ be a finite dimensional Frobenius algebra with 1 over $\mathbf{k}$. Let $\{e_a\}$ be a basis of $H$ as above. Denote by $\{x^a\}$ the linear coordinates in the basis $\{e_a\}$. Consider a self-parameterizing family $(H, \{\cdot_x, x \in U\})$, where $U$ is an open subset of $H$, such that 1 is the identity for each $\wedge_x$ and

$$(X \wedge_x Y, Z) = (X, Y \wedge_x Z),$$

for all $X, Y, Z \in H$ and $x \in U$. Then we get a family of 3-tensors $\phi_{abc}(x)$. String theory (see e.g. Dijkgraaf-Verlinde-Verlinde [10]) suggests that one should require

$$\frac{\partial}{\partial x^d}\phi_{abc} = \frac{\partial}{\partial x^c}\phi_{abd}.$$

Under this condition, if $U$ is contractible, one can find a function $\Phi : U \to \mathbf{k}$, such that

$$\phi_{abc}(x) = \frac{\partial^3 \Phi}{\partial x^a \partial x^b \partial x^c}.$$

By (2), the associativity of $\wedge_x$ is then equivalent to that $\Phi$ satisfies the following Witten-Dijkgraaf-E. Verlinde-H. Verlinde (WDVV) equations:

$$\frac{\partial^3 \Phi}{\partial x^a \partial x^b \partial x^p}\eta^{pq}\frac{\partial^3 \Phi}{\partial x^q \partial x^c \partial x^d} = \frac{\partial^3 \Phi}{\partial x^b \partial x^c \partial x^p}\eta^{pq}\frac{\partial^3 \Phi}{\partial x^a \partial x^q \partial x^d}. \tag{3}$$

Such a function $\Phi$ is called a *potential function*. Dubrovin [11, 12] introduced the notion of a Frobenius manifold to give a global formulation. For our purpose in this paper, a Frobenius manifold structure on $(H, \wedge, (\cdot,\cdot))$ will mean a solution $\Phi$ to the WDVV equations with

$$\eta_{ab} = \frac{\partial^3 \Phi}{\partial x^0 \partial x^a \partial x^b}. \tag{4}$$

By definition a structure of *formal Frobenius manifold* on $(H, \wedge, (\cdot,\cdot))$ is a formal power series $\Phi$ which satisfies the WDVV equations. We refer to $(H, \wedge, (\cdot,\cdot))$ as the initial data for the WDVV equations. If $\Phi$ also satisfies (4), it is called a structure of formal Frobenius manifold with identity. The above discussion can be carried out for graded algebras as well. See Manin [22].

1.3. **DGBV algebras and Frobenius manifolds.** Let $(\mathcal{A}, \wedge)$ be a graded commutative associative algebra over $\mathbf{k}$. For any linear operator $\Delta$ of odd degree, define

$$[a \bullet b]_\Delta = (-1)^{|a|}(\Delta(a \wedge b) - (\Delta a) \wedge b - (-1)^{|a|} a \wedge \Delta b),$$

for homogeneous elements $a, b \in \mathcal{A}$. If $\Delta^2 = 0$ and

$$[a \bullet (b \wedge c)]_\Delta = [a \bullet b]_\Delta \wedge c + (-1)^{(|a|+1)|b|} b \wedge [a \bullet c]_\Delta,$$

for all homogeneous $a, b, c \in \mathcal{A}$, then $(\mathcal{A}, \wedge, \Delta, [\cdot \bullet \cdot]_\Delta)$ is a *Gerstenhaber-Batalin-Vilkovisky (GBV) algebra*. (Notice that if one takes $a = b = c = 1$, then one can deduce $\Delta 1 = 0$.) A *DGBV (differential Gerstenhaber-Batalin-Vilkovisky) algebra* is a GBV algebra with a $\mathbf{k}$-linear derivation $\delta$ of odd degree with respect to $\wedge$, such that

$$\delta^2 = \delta\Delta + \Delta\delta = 0.$$

We will be interested in the cohomology group $H(\mathcal{A}, \delta)$. A $\mathbf{k}$-linear functional $\int : \mathcal{A} \to \mathbf{k}$ on a DGBV-algebra is called *an integral* if for all $a, b \in \mathcal{A}$,

$$\int (\delta a) \wedge b = (-1)^{|a|+1} \int a \wedge \delta b, \tag{5}$$

$$\int (\Delta a) \wedge b = (-1)^{|a|} \int a \wedge \Delta b. \tag{6}$$

Under these conditions, it is clear that $\int$ induces a scalar product on $H = H(\mathcal{A}, \delta)$: $(a, b) = \int a \wedge b$. If it is nondegenerate on $H$, we say that the integral is *nice*. It is obvious that

$$(\alpha \wedge \beta, \gamma) = (\alpha, \beta \wedge \gamma).$$

Hence if $\mathcal{A}$ has a nice integral, $(H, \wedge, (\cdot, \cdot))$ is a (graded) Frobenius algebra.

Under suitable conditions, one can construct Frobenius manifolds from DGBV algebras. The following result is due to Barannikov-Kontsevich [2] and Manin [23]:

**Theorem 1.1.** *Let $(\mathcal{A}, \wedge, \delta, \Delta, [\cdot \bullet \cdot])$ be a DGBV algebra satisfying the following conditions:*

(a) *$H = H(\mathcal{A}, \delta)$ is finite dimensional.*
(b) *There is a nice integral on $\mathcal{A}$.*
(c) *The inclusions $(Ker\Delta, \delta) \hookrightarrow (\mathcal{A}, \delta)$ and $(Ker\delta, \Delta) \hookrightarrow (\mathcal{A}, \Delta)$ induce isomorphisms of cohomology.*

*Then there is a canonical construction of a formal Frobenius manifold structure with identity on $H$.*

We now indicate how to obtain the potential function $\Phi$. It is based on the existence of a solution $\Gamma = \sum \Gamma_n$ to

$$\delta\Gamma + \tfrac{1}{2}[\Gamma \bullet \Gamma] = 0,$$
$$\Delta\Gamma = 0,$$

which satisfies the following conditions: (a) $\Gamma_0 = 0$; (b) $\Gamma_1 = \sum x^j e_j$, $e_j \in \operatorname{Ker} \delta \cap \operatorname{Ker} \Delta$, where the classes of $e_j$'s generates $H = H(\mathcal{A}, \delta)$; (c) for $n > 1$, $\Gamma_n \in \operatorname{Im} \Delta$ is a homogeneous super polynomial of degree $n$ in $x^j$'s, such that the total degree of $\Gamma_n$ is even; (d) $x^0$ only appears in $\Gamma_1$. Such a solution is called a *normalized universal solution*. Under suitable conditions, its existence can be established inductively. Let $\Gamma = \Gamma_1 + \Delta B$ be a normalized solution, then

$$\Phi = \int \frac{1}{6}\Gamma^3 - \frac{1}{2}\delta B \Delta B = \int \frac{1}{6}\Gamma^3 - \frac{1}{4}\Gamma \wedge \Gamma \wedge (\Gamma - \Gamma_1).$$

## 2. Cartan model of equivariant cohomology

We will use the Cartan model for equivariant cohomology. We refer the readers to Atiyah-Bott [1] and Berline-Getzler-Vergne [4] for more details. Throughout this paper, $K$ will be a compact connected Lie group, with $\mathfrak{k}$ as its Lie algebra. Let $M$ be a compact smooth $K$-manifold. The $K$-action on $M$ induces a homomorphism from the Lie algebra $\mathfrak{k}$ to the Lie algebra of vector fields on $M$. Let $\{\xi_a\}$ be a basis of $\mathfrak{k}$, such that

$$[\xi_a, \xi_b] = f_{ab}^c \xi_c,$$

where $f_{ab}^c$'s are the structure constants. Let $\{\Theta^a\}$ be the dual basis in $\mathfrak{k}^*$. Denote by $\iota_a$ and $\mathcal{L}_a$ the contraction and the Lie derivative by the vector field corresponding to $\xi_a \in \mathfrak{k}$ respectively. The Cartan model is given by the complex $(\Omega_K(X), D_K)$,

where $\Omega_K(M) = (S(\mathfrak{k}^*) \otimes \Omega(M))^K$, and $D_K = 1 \otimes d - \Theta^a \otimes \iota_a$, which is called the Cartan differential. Since $D_K$ is a $K$-invariant operator on $S(\mathfrak{k}^*) \otimes \Omega(M)$, it then maps $\Omega_K(M)$ to itself. Furthermore, since $\Theta^a L_a$ acts as zero on $S(\mathfrak{k}^*)$, we have

$$D_K^2 = -\Theta^a \otimes \mathcal{L}_a = -\Theta^a(L_a \otimes 1 + 1 \otimes \mathcal{L}_a).$$

Therefore, $D_K^2 = 0$ on $\Omega_K(M)$. The Cartan model defines equivariant cohomology of the $K$-manifold $M$ as

$$H_K^*(M) = \operatorname{Ker} D_K / \operatorname{Im} D_K.$$

The wedge product $\wedge$ on $\Omega^*(M)$ can be extended to $\Omega_K^*(M)$. This makes $\Omega_K^*(M)$ an algebra over $S(\mathfrak{k}^*)^K$. It is easy to see that $D_K$ is a derivation, i.e.,

$$D_K(\alpha \wedge \beta) = (D_K\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge D_K\beta,$$

for homogeneous $\alpha, \beta \in \Omega_K^*(M)$. Hence $H_K^*(M)$ is an algebra over $S(\mathfrak{k}^*)^K$.

Notice that there is a $S(\mathfrak{k}^*)^K$-linear operator

$$\int_M : \Omega_K^*(M) \to S(\mathfrak{k}^*)^K$$

which is defined by sending differential forms of degree $\dim(M)$ to its integral over $M$, and all other forms to zero. Since we assume $M$ has no boundary, by Stokes theorem, it is easy to see that

$$\int_M (D_K\alpha) \wedge \beta = (-1)^{|\alpha|+1} \int_M \alpha \wedge D_K\beta. \tag{7}$$

For simplicity of notation, we will simply write $D_K = d - C$, where $C = \Theta^a \iota_a$. Then $dC + Cd = 0$, $C^2 = 0$ on $\Omega_K^*(M)$. There is a natural bigrading on $\Omega_K^*(M)$:

$$(\Omega_K^*(M))^{p,q} = (\Omega^{p-q}(M) \otimes S^q(\mathfrak{k}^*))^K.$$

With respect to this bigrading, $d$ has bidegree $(1, 0)$, $C$ has bigrading $(0, 1)$. Every element $\alpha_K \in \Omega_K^*(M)$ can be written as

$$\alpha_K = \sum_{k \geq 0} \alpha^{(2k)},$$

such that $D_K\alpha_K = 0$ if and only if $d\alpha^{(0)} = 0$ and $d\alpha^{(2k+2)} = C\alpha^{(2k)}$, $k \geq 0$.

## 3. Invariant Poisson structure and DGBV algebra structure on Cartan model

We now assume that $M$ has an $K$-invariant Poisson structure $w$, i.e. $w \in \Gamma(M, \Lambda^2 TM)^K$ and the Schouten-Nijenhuis bracket $[w, w] = 0$. For any Poisson structure $w$, Koszul [18] defined an operator $\Delta : \Omega^*(M) \to \Omega^{*-1}(M)$ by $\Delta = [\iota_w, d]$. He also showed that $\Delta$ has the following important properties:

$$\Delta^2 = 0, \quad [d, \Delta] = d\Delta + \Delta d = 0,$$

and if we set

$$[\alpha, \beta]_\Delta = (-1)^{|\alpha|}(\Delta(\alpha \wedge \beta) - (\Delta\alpha) \wedge \beta - (-1)^{|\Delta|}\alpha \wedge \Delta\beta),$$

then

$$[\alpha, \beta \wedge \gamma]_\Delta = [\alpha, \beta \wedge \gamma]_\Delta \wedge \gamma + (-1)^{(|\alpha|+1)|\beta|}\beta \wedge [\alpha, \gamma]_\Delta.$$

This implies that $(\Omega^*(M), \wedge, d, \Delta, [\cdot,\cdot]_\Delta)$ is a DGBV algebra. In our case, we can extend $\Delta$ and $[\cdot,\cdot]_\Delta$ to $\Omega^*(M) \otimes S(\mathfrak{k}^*)$. It is clear that they both commute with the group action. Hence they restrict to $\Omega_K^*(M)$.

**Proposition 3.1.** *Let $M$ be a $K$-manifold with a $K$-invariant Poisson structure $w$, then $(\Omega^*_K(M), \wedge, D_K, \Delta, [\cdot,\cdot]_\Delta)$ is a DGBV algebra.*

*Proof.* One only needs to prove $[D_K, \Delta] = 0$. Now $D_K = d - C = d - \Theta^a \iota_a$. We have $[d, \Delta] = 0$, and

$$[\iota_a, \Delta] = [\iota_a, [\iota_w, d]] = [[\iota_a, \iota_w], d] + [\iota_w, [\iota_a, d]] = [\iota_w, \mathcal{L}_a] = -[\iota_w, L_a] = 0$$

on $\Omega^*_K(M)$, hence $[C, \Delta] = 0$. The proof is complete. □

From [7], we also have

**Lemma 3.1.** *For a $K$-manifold $M$ with a $K$-invariant Poisson structure $w$, we have*

$$\int_M (\Delta\alpha) \wedge \beta = (-1)^{|\alpha|} \int \alpha \wedge \Delta\beta.$$

## 4. Symplectic manifolds with Hamiltonian actions

We now assume that $M$ has a symplectic structure $\omega$ and the $K$-action is Hamiltonian, i.e., the $K$-action preserves $\omega$ and there is a $K$-equivariant map $\mu : M \to \mathfrak{k}^*$, such that

$$d\langle \mu, \xi_a \rangle = \iota_a \omega.$$

The symplectic structure $\omega$ induces an isomorphism $T^*M \cong TM$, hence isomorphisms $\Omega^*(M) \cong \Gamma(M, \Lambda^* TM)$. Denote by $w$ the bi-vector field corresponding to $\omega$. Then $w$ is an invariant Poisson structure. Hence, we have

**Proposition 4.1.** *For a symplectic manifold $M$ with a Hamiltonian $K$-action, $(\Omega^*_K(M), \wedge, D_K, \Delta, [\cdot,\cdot]_\Delta)$ is a DGBV algebra.*

*Remark* 4.1. It is tempting to define $\Delta_K = \Delta - d\mu\wedge$. Indeed, it is easy to show that $\Delta_K^2 = [D_K, \Delta_K] = 0$. However, we do not have

$$[\alpha, \beta \wedge \gamma]_{\Delta_K} = [\alpha, \beta \wedge \gamma]_{\Delta_K} \wedge \gamma + (-1)^{(|\alpha|+1)|\beta|} \beta \wedge [\alpha, \gamma]_{\Delta_K}.$$

Hence $(\Omega^*_K(M), \wedge, D_K, \Delta_K, [\cdot,\cdot]_{\Delta_K})$ is not a DGBV algebra.

For a closed symplectic manifold $M$ with a Hamiltonian $K$-action, a result of Kirwan [17] (p. 68, Proposition 5.8) says that $H^*_K(M) \cong H^*(M) \otimes_{\mathbb{R}} S(\mathfrak{k}^*)^K$ as vector spaces over $\mathbb{R}$. An important consequence of the above result of Kirwan is that every de Rham cohomology class of $M$ has a representative $\alpha$, which can be extended to a $D_K$ closed form $\alpha_K$ of the form

$$\alpha_K = \alpha + \Theta^a \alpha_a + \cdots .$$

Therefore, one can find $D_K$-closed forms $\{\alpha_{Ki} = \alpha_i + \Theta^a \alpha_{ia} + \cdots\}$ such that the matrix $(\int_M \alpha_i \wedge \alpha_j)$ is invertible over $\mathbb{R}$. However the matrix $(\int_M \alpha_{Ki} \wedge \alpha_{Kj})$ may not be invertible over $S(\mathfrak{k}^*)^K$. Later we will prove that for Hamiltonian actions on a closed Kähler manifold by holomorphic isometries, we can find natural extensions $\{\alpha_{Ki}\}$ such that

$$\int_M \alpha_{Ki} \wedge \alpha_{Kj} = \int_M \alpha_i \wedge \alpha_j.$$

For now, to invert the matrix $(\int_M \alpha_{Ki} \wedge \alpha_{Kj})$, we need to work over a field. Denote by $T$ a maximal torus of $K$, $\mathfrak{t}$ its Lie algebra and $W$ the Weyl group. Then $S(\mathfrak{k}^*)^K = S(\mathfrak{t}^*)^W$. Hence $S(\mathfrak{k}^*)^K$ is an integral domain, since $S(\mathfrak{t}^*)$ is a polynomial

algebra. Denote by $F(\mathfrak{k}^*)$ its fractional field, i.e. $F(\mathfrak{k}^*) = \{f/g : f, g \in S(\mathfrak{k}^*)^K\}$. Define

$$\widetilde{\Omega}^*_K(M) = \Omega^*_K(M) \otimes_{S(\mathfrak{k}^*)^K} F(\mathfrak{k}^*),$$

Extend $D_K$, $\wedge$, $\Delta$ etc. to $\widetilde{\Omega}^*_K(M)$ and define

$$\widetilde{H}^*_K(M) = H^*(\widetilde{\Omega}^*_K(M), D_K).$$

Then we have

$$\widetilde{H}^*_K(M) = H^*_K(M) \otimes_{S(\mathfrak{k}^*)^K} F(\mathfrak{k}^*)$$

as vector spaces over $F(\mathfrak{k}^*)$. Now the matrix $(\int_M \alpha_{Ki} \wedge \alpha_{Kj})$ has a nonzero determinant, hence it is invertible over $F(\mathfrak{k}^*)$. Therefore $(\widetilde{\Omega}^*_K(M), \wedge, D_K, \Delta, [\cdot,\cdot]_\Delta)$ satisfies conditions (a) and (b) in Theorem 1.1 over the field $\mathbf{k} = F(\mathfrak{k}^*)$. Thus we have the following

**Theorem 4.1.** *Let $M$ be a closed symplectic manifold with a Hamiltonian $K$-action. Suppose that the inclusions $i : (\operatorname{Ker}\Delta, D_K) \hookrightarrow (\widetilde{\Omega}^*_K(M), D_K)$ and $j : (\operatorname{Ker} D_K, \Delta) \hookrightarrow (\widetilde{\Omega}^*_K(M), \Delta)$ induce isomorphisms on cohomology. Then over the field $F(\mathfrak{k}^*)$, the DGBV algebra $(\widetilde{\Omega}^*_K(M), \wedge, D_K, \Delta, [\cdot,\cdot]_\Delta)$ satisfies all the conditions in Theorem 1.1. Hence there is a canonical construction of formal Frobenius manifold structure on $\widetilde{H}^*_K(M)$.*

## 5. Kähler manifolds with holomorphic Hamiltonian actions

We now further restrict our attention to a closed Kähler manifold $M$, such that $K$ acts on $M$ by holomorphic isometries. Then the Kähler form is an invariant symplectic form, hence the results in §3 apply. The main advantage here is that for a Kähler manifold, we can exploit some nice features of the Hodge theory to establish the quasi-isomorphisms property (c) in Theorem 1.1. Note that in Lemma 5.2-5.4, we will not require the $K$-action to be Hamiltonian.

The almost complex structure $J : TM \to TM$ induces a decomposition $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$. There is an induced decomposition $\Omega^*(M) \otimes_{\mathbb{R}} \mathbb{C} = \Omega^{*,*}(M)$, and $d = \partial + \bar{\partial}$, where

$$\partial : \Omega^{*,*}(M) \to \Omega^{*+1,*}(M), \quad \bar{\partial} : \Omega^{*,*}(M) \to \Omega^{*,*+1}(M).$$

Denote also by $J$ the linear map on $\Omega^{*,*}(M)$ induced by $J$. Then we have

$$J\alpha = (-1)^q i^{p+q} \alpha,$$

for $\alpha \in \Omega^{p,q}(M)$. Hence $J^2 = (-1)^{p+q}$ and $J^{-1} = (-1)^p i^{p+q}$ on $\Omega^{p,q}(M)$. It is easy to see that

$$J^{-1}\partial J = -i\partial, \quad J^{-1}\bar{\partial} J = i\bar{\partial}.$$

Hence

$$J^{-1} dJ = J^{-1}\partial J + J^{-1}\bar{\partial} J = i(\bar{\partial} - \partial) = d^c.$$

We are also interested in the formal adjoints $d^*$, $\partial^*$ and $\bar{\partial}^*$. Since

$$\partial^* : \Omega^{*,*}(M) \to \Omega^{*-1,*}(M), \quad \bar{\partial}^* : \Omega^{*,*}(M) \to \Omega^{*,*-1}(M).$$

Hence

$$J^{-1}\partial^* J = i\partial^*, \quad J^{-1}\bar{\partial}^* J = -i\bar{\partial}^*.$$

Therefore

$$J^{-1} d^* J = J^{-1}\partial^* J + J^{-1}\bar{\partial}^* J = -i(\bar{\partial}^* - \partial^*) = (d^c)^*.$$

With the help of Kähler identities, one can obtain the following well-known equalities (see e.g. Deligne-Griffiths-Morgan-Sullivan [9]):

$$\begin{aligned} &d^2 = (d^c)^2 = dd^c + d^c d = 0, \\ &(d^*)^2 = ((d^c)^*)^2 = d^*(d^c)^* + (d^c)^* d^* = 0, \\ &d(d^c)^* + (d^c)^* d = d^* d^c + d^c d^* = 0, \\ &dd^* + d^* d = d^c (d^c)^* + (d^c)^* d^c = \Box. \end{aligned}$$

Here, $\Box$ in the last equality denotes the Laplace operator on forms. As a consequence, one has the following Hodge decompositions

$$\begin{aligned} \Omega^*(M) &= \mathcal{H} \oplus \operatorname{Im} d \oplus \operatorname{Im} d^* = \mathcal{H} \oplus \operatorname{Im} d^c \oplus \operatorname{Im}(d^c)^* \\ &= \mathcal{H} \oplus \operatorname{Im} dd^c \oplus \operatorname{Im} d^* d^c \oplus \operatorname{Im} d(d^c)^* \oplus \operatorname{Im} d^*(d^c)^*, \end{aligned}$$

where $\mathcal{H}$ is the space of harmonic forms.

**Lemma 5.1.** *On a closed Kähler manifold $M$, if $\Delta d\beta = 0$ for some $\beta \in \Omega_K^*(M)$, then there exist $\beta^H \in \operatorname{Ker} \Box$, $a, b, c \in \Omega_K^*(M)$, such that*

$$\beta = \beta^H + \Delta^* da + \Delta db + \Delta d^* c.$$

*Proof.* It suffices to prove the result for $\Omega^*(M)$. The extension to $\Omega_K^*(M)$ is straightforward. But for $\Omega^*(M)$, Käkler identity implies that $\Delta = -(d^c)^*$, so the lemma follows from the above five-fold decomposition. $\Box$

**Lemma 5.2.** *The inclusion $j : (\operatorname{Ker} D_K, \Delta) \hookrightarrow (\Omega_K^*(M), \Delta)$ induces an injective map in cohomology.*

*Proof.* We need to show that if $D_K \alpha_K = 0$ and $\alpha_K = \Delta\beta_K$ for some $\beta_K \in \Omega_K^*(M)$, then there exists $\beta_K' \in \operatorname{Ker} D_K$, such that $\alpha_K = \Delta\beta_K'$. We use the bigrading on $\Omega_K^*(M)$ to write

$$\alpha_K = \sum_{k \geq 0} \alpha^{(2k)}, \quad \beta_K = \textstyle\sum_{k \geq 0} \beta^{(2k)},$$

such that

$$\begin{aligned} &d\alpha^{(0)} = 0, && \alpha^{(0)} = \Delta\beta^{(0)}, \\ &d\alpha^{(2)} = C\alpha^{(0)}, && \alpha^{(2)} = \Delta\beta^{(2)}, \\ &\cdots\cdots \end{aligned}$$

We will repeatedly use the following corollary of Lemma 5.1: if $d\Delta\beta = 0$ for some $\beta \in \Omega_K^*(M)$, then there exists $\gamma \in \Omega_K^*(M)$, such that

$$\Delta\beta = \Delta d\gamma.$$

In fact, we can take $\gamma = Gd^*\beta$, where $G$ is the Green operator, so that

$$\Delta d\gamma = \Delta dGd^*\beta = Gdd^*\Delta\beta = G(\Box - d^*d)\Delta\beta = G\Box\Delta\beta = \Delta\beta.$$

Now $d\Delta\beta^{(0)} = d\alpha^{(0)} = 0$, hence

$$\alpha^{(0)} = \Delta\beta^{(0)} = \Delta d\gamma^{(0)},$$

where $\gamma^{(0)} = Gd^*\beta^{(0)}$. Also,

$$d\Delta\beta^{(2)} = d\alpha^{(2)} = C\alpha^{(0)} = C\Delta d\gamma^{(0)} = -d\Delta C\gamma^{(0)}.$$

Hence $\Delta d(\beta^{(2)} + C\gamma^{(0)}) = 0$. Therefore,

$$\Delta(\beta^{(2)} + C\gamma^{(0)}) = \Delta d\gamma^{(2)},$$

where $\gamma^{(2)} = Gd^*(\beta^{(2)} + C\gamma^{(0)})$. Equivalently, we have

$$\alpha^{(2)} = \Delta\beta^{(2)} = \Delta d\gamma^{(2)} - \Delta C\gamma^{(0)} = \Delta(d\gamma^{(2)} - C\gamma^{(0)}).$$

Inductively, we have for $k \geq 0$,

$$\alpha^{(2k+2)} = \Delta(d\gamma^{(2k+2)} - C\gamma^{(2k)}),$$

where $\gamma^{(2k+2)} = Gd^*(\beta^{(2k+2)} + C\gamma^{(2k)})$. Setting $\beta'_K = D_K \sum_{k\geq 0} \gamma^{(2k)}$, it is then clear that $D_K\beta'_K = 0$ and

$$\alpha_K = \Delta\beta_K = \Delta\beta'_K.$$

□

*Remark* 5.1. We actually prove the following stronger result:

$$\operatorname{Ker} D_K \cap \operatorname{Im}\Delta = \operatorname{Im}\Delta D_K.$$

**Lemma 5.3.** *The inclusion* $i : (\operatorname{Ker}\Delta, D_K) \hookrightarrow (\Omega^*_K(M), D_K)$ *induces an injective map in cohomology.*

*Proof.* We need to show that if $\Delta\alpha_K = 0$ and $\alpha_K = D_K\beta_K$ for some $\beta_K \in \Omega^*_K(M)$, then there exists $\beta'_K \in \operatorname{Ker}\Delta$, such that $\alpha_K = D_K\beta'_K$. We will use repeatedly the following corollary of Lemma 5.1: if $d\Delta\beta = 0$ for some $\beta \in \Omega^*_K(M)$, then there exist $\beta^H \in \operatorname{Ker}\Box$, $\phi, \psi \in \Omega^*_K(M)$, such that

$$\beta = \beta^H + \Delta\phi + d\psi.$$

Decompose $\alpha_K$ and $\beta_K$ as usual. We have

$$\begin{aligned} &\Delta\alpha^{(0)} = 0, && \alpha^{(0)} = d\beta^{(0)}, \\ &\Delta\alpha^{(2)} = 0, && \alpha^{(2)} = d\beta^{(2)} - C\beta^{(0)}, \\ &\cdots\cdots \end{aligned}$$

Now $\Delta d\beta^{(0)} = \Delta\alpha^{(0)} = 0$, hence

$$\beta^{(0)} = (\beta^{(0)})^H + \Delta\phi^{(0)} + d\psi^{(0)}.$$

Therefore,

$$\Delta d\beta^{(2)} = \Delta(\alpha^{(2)} + C\beta^{(0)}) = -C\Delta\beta^{(0)} = -C\Delta d\psi^{(0)} = -\Delta dC\psi^{(0)}.$$

So we have $\Delta d(\beta^{(2)} + C\psi^{(0)}) = 0$, hence

$$\beta^{(2)} + C\psi^{(0)} = (\beta^{(2)} + C\psi^{(0)})^H + \Delta\phi^{(2)} + d\psi^{(2)}.$$

By induction, we find that for $k \geq 0$,

$$\beta^{(2k+2)} = (\beta^{(2k+2)} + C\psi^{(2k)})^H + \Delta\phi^{(2k+2)} + d\psi^{(2k+2)} - C\psi^{(2k)}.$$

Setting $\phi_K = \sum_{k\geq 0} \phi^{(2k)}$ and $\psi_K = \sum_{k\geq 0} \psi^{(2k)}$, then we have

$$\beta_K = (\beta_K + C\psi_K)^H + \Delta\phi_K + D_K\psi_K.$$

Hence

$$\alpha_K = D_K\beta_K = D_K((\beta_K + C\psi_K)^H + \Delta\phi_K). \tag{8}$$

Since $\Delta((\beta_K + C\psi_K)^H + \Delta\phi_K) = 0$, the proof is complete. □

*Remark* 5.2. We actually have proved $\operatorname{Ker}\Delta \cap \operatorname{Im} D_K = D_K \operatorname{Ker}\Delta$. If one can show that for any $\beta \in \operatorname{Ker}\Box$, there exists $\gamma \in \Omega^*_K(M)$, such that $D_K(\beta + \Delta\gamma) = 0$, then by (8), $\operatorname{Ker}\Delta \cap \operatorname{Im} D_K = \operatorname{Im} D_K \Delta$.

**Lemma 5.4.** *The inclusion* $i : (\operatorname{Ker}\Delta, D_K) \hookrightarrow (\Omega^*_K(M), D_K)$ *induces a surjective map in cohomology.*

*Proof.* We need to show that If $D_K\alpha_K = 0$ for some $\alpha_K \in \Omega^*_K(M)$, then there exists $\beta_K \in \Omega^*_K(M)$, such that $\Delta(\alpha_K - D_K\beta_K) = 0$. Decompose $\alpha_K$ as usual, then we have $d\alpha^{(0)} = 0$, $d\alpha^{(2)} = C\alpha^{(0)}$, etc. First of all, there exists $\beta^{(0)} \in \Omega^*_K(M)$, such that $\alpha^{(0)} - d\beta^{(0)} = (\alpha^{(0)})^H \in \operatorname{Ker}\Box$. Now

$$\Delta d(\alpha^{(2)} + C\beta^{(0)}) = \Delta C\alpha^{(0)} - \Delta C d\beta^{(0)} = \Delta C(\alpha^{(0)})^H = 0.$$

Hence $\alpha^{(2)} + C\beta^{(0)} = (\alpha^{(2)} + C\beta^{(0)})^H + \Delta\gamma^{(2)} + d\beta^{(2)}$. By induction, we can find $\beta^{(2k+2)} \in \Omega^*_K(M)$, for $k \geq 0$, such that

$$\alpha^{(2k+2)} = (\alpha^{(2k+2)} + C\beta^{(2k)})^H + \Delta\gamma^{(2k+2)} + d\beta^{(2k+2)} - C\beta^{(2k)}.$$

Set $\beta_K = \sum_{k\geq 0} \beta^{(2k)}$, $\gamma_K = \sum_{k\geq 1} \gamma^{(2k)}$. Then we have

$$\alpha_K = (\alpha_K + C\beta_K)^H + \Delta\gamma_K + D_K\beta_K.$$

Hence

$$\Delta(\alpha_K - D_K\beta_K) = \Delta((\alpha_K + C\beta_K)^H + \Delta\gamma_K) = 0.$$

□

From now on we assume that the $K$-action is Hamiltonian, and let $\mu = \Theta^a\mu_a$ be the moment map.

**Lemma 5.5.** $\iota_a\alpha = \mu_a\Delta\alpha - \Delta(\mu_a\alpha)$.

*Proof.* For $\alpha, \beta \in \Omega^*(M)$, we have

$$\begin{aligned}
& \langle \iota_a\alpha, \beta\rangle = \langle \alpha, Jd\mu_a \wedge \beta\rangle = \langle J\alpha, J(Jd\mu_a \wedge \beta)\rangle \\
= \quad & -\langle J\alpha, d\mu_a \wedge J\beta\rangle = -\langle J\alpha, d(\mu_a J\beta) - \mu_a dJ\beta\rangle \\
= \quad & -\langle d^*J\alpha, \mu_a J\beta\rangle + \langle \mu_a J\alpha, dJ\beta\rangle \\
= \quad & -\langle \mu_a d^*J\alpha, J\beta\rangle + \langle d^*J(\mu_a\alpha), J\beta\rangle \\
= \quad & -\langle J^{-1}(\mu_a d^*J\alpha) - J^{-1}d^*J(\mu_a\alpha), \beta\rangle \\
= \quad & -\langle \mu_a(d^c)^*\alpha) - (d^c)^*(\mu_a\alpha), \beta\rangle \\
= \quad & \langle \mu_a\Delta\alpha - \Delta(\mu_a\alpha), \beta\rangle.
\end{aligned}$$

□

**Corollary 5.1.** *If* $\Delta\alpha = 0$*, then* $C(\alpha) = -\Delta(\mu\alpha)$.

**Proposition 5.1.** *On a closed Kähler manifold* $M$ *with a Hamiltonian* $K$*-action by holomorphic isometries, any harmonic form* $\alpha^{(0)}$ *can be canonically extended to a* $D_K$*-closed form* $\sum_{k\geq 0} \alpha^{(2k)}$*, where* $\alpha^{(2k+2)} = -Gd^*\Delta(\mu\alpha^{(2k)}) \in \operatorname{Im}\Delta d^*$.

*Proof.* First notice that $\alpha^{(0)} \in \Omega^*(M)^K$. Indeed, since $K$ is connected, the action of $K$ on $H^*(M)$ is trivial. Hence for any $g \in G$, $g(\alpha^{(0)})$ is a harmonic form in the same cohomology class as $\alpha^{(0)}$, therefore, $g(\alpha^{(0)}) = \alpha^{(0)}$. By Corollary 5.1, $C(\alpha^{(0)}) = -\Delta(\mu\alpha^{(0)})$. Since $dC(\alpha^{(0)}) = -Cd\alpha^{(0)} = 0$ and $C(\alpha^{(0)})$ has no harmonic part, we have

$$C(\alpha^{(0)}) = d\alpha^{(2)}$$

where $\alpha^{(2)} = -Gd^*\Delta(\mu\alpha^{(0)}) = Gd^*C(\alpha^{(0)}) \in \Omega^*_K(M)$. Now since $\alpha^{(2)} \in \operatorname{Im}\Delta$, $C(\alpha^{(2)}) = -\Delta(\mu\alpha^{(2)})$. From

$$dC(\alpha^{(2)}) = -Cd\alpha^{(2)} = -C^2(\alpha^{(0)}) = 0,$$

we see that if we set $\alpha^{(4)} = -Gd^*\Delta(\mu\alpha^{(2)})$, then $\alpha^{(4)} \in \Omega^*_K(M)$ and we have $d\alpha^{(4)} = C(\alpha^{(2)})$. Inductively, one gets $\alpha^{(2k)}$ in the same way. This process terminates after finitely many steps, since each time the degree of the differential forms are reduced by 2. Then $\sum_{k\geq 0} \alpha^{(2k)}$ is a $D_K$-closed form. □

As a corollary, we get an easy proof of

$$H^*_K(M) \cong H^*(M) \otimes S(\mathfrak{k}^*)^K$$

in the case of closed Kähler manifolds. As another corollary, we get the following

**Lemma 5.6.** *On a closed Kähler manifold $M$ with a Hamiltonian $K$-action by holomorphic isometries, the inclusion $j : (\operatorname{Ker} D_K, \Delta) \hookrightarrow (\Omega^*_K(M), \Delta)$ induces a surjective map in cohomology.*

*Proof.* It suffices to show that for any $\alpha^{(0)} \in \operatorname{Ker}\Delta$, there exists $\beta \in \Omega^*_K(M)$, such that $D_K(\alpha^{(0)} + \Delta\beta) = 0$. Without loss of generality, we can assume that $\alpha^{(0)}$ is harmonic. By Proposition 5.1, we can take

$$\beta = Gd^* \sum_{k\geq 0} \mu\alpha^{(2k)},$$

where $\alpha^{(2k+2)} = -Gd^*\Delta(\mu\alpha^{(2k)})$. □

Combining Lemmas 5.2, 5.3, 5.4 and 5.6, we get

**Theorem 5.1.** *On a closed Kähler manifold $M$ with a Hamiltonian $K$-action by holomorphic isometries, the inclusions $i : (\operatorname{Ker}\Delta, D_K) \hookrightarrow (\Omega^*_K(M), D_K)$ and $j : (\operatorname{Ker} D_K, \Delta) \hookrightarrow (\Omega^*_K(M), \Delta)$ induce isomorphisms in cohomology.*

## 6. Normalized universal solution and formal Frobenius manifold structure

Assume that $\{\omega_a^{(0)} \in \mathcal{H}\}$ gives rise to a homogeneous basis of $H^*(M)$. By Proposition 5.1, each $\omega_a^{(0)}$ can be extended to a $D_K$-closed class

$$\omega_{Ka} = \sum_{k\geq 0} \omega_a^{(2k)},$$

where $\alpha_a^{(2k+2)} = -Gd^*\Delta(\mu\alpha_a^{(2k)})$ for $k \geq 0$. In particular, $\alpha_{K0} = 1$. Now $\{\omega_{Ka}\}$ are free generators of $H^*_K(M)$. Write $\omega_{Ka} = \omega_a^{(0)} + \Delta\gamma_a$, then from Lemma 3.1 we

have

$$\begin{aligned}
&\int_M \omega_{Ka} \wedge \omega_{Kb} = \int_M (\omega_a^{(0)} + \Delta\gamma_a) \wedge (\omega_b^{(0)} + \Delta\gamma_b) \\
= &\int_M \omega_a^{(0)} \wedge \omega_b^{(0)} + \Delta\gamma_a \wedge \omega_b^{(0)} + \omega_a^{(0)} \wedge \Delta\gamma_b + \Delta\gamma_a \wedge \Delta\gamma_b \\
= &\int_M \omega_a^{(0)} \wedge \omega_b^{(0)} \pm \gamma_a \wedge \Delta\omega_b^{(0)} \pm \Delta\omega_a^{(0)} \wedge \gamma_b \pm \Delta^2\gamma_a \wedge \gamma_b \\
= &\int_M \omega_a^{(0)} \wedge \omega_b^{(0)}.
\end{aligned}$$

In other words, the matrix $(\eta_{ab})$ for the pairing $(\cdot,\cdot)$ is the same as in the ordinary case. Hence we can take $\mathbf{k} = S(\mathfrak{k}^*)^K$ in Theorem 1.1. Thus, we have

**Theorem 6.1.** *Let $M$ be a closed Kähler manifold with a Hamiltonian $K$-action by holomorphic isometries, then there is a formal Frobenius manifold structure on $H_K^*(M)$ obtained by DGBV algebraic construction over $S(\mathfrak{k}^*)^K$.*

To obtain the normalized universal solution, we take $\Gamma_{K1} = x^a\alpha_{Ka}$. For $n > 1$, we find $\Gamma_{Kn} \in \operatorname{Im}\Delta$ by inductively solving

$$D_K\Gamma_{Kn} = -\frac{1}{2}\sum_{p=1}^{n-1}[\Gamma_{Kp} \bullet \Gamma_{Kn-p}]_\Delta = -\frac{1}{2}\sum_{p=1}^{n-1}\Delta(\Gamma_{Kp} \wedge \Gamma_{Kn-p}).$$

Standard argument shows that the right hand side is $D_K$-closed, hence by the proof of Lemma 5.2, we take

$$\Gamma_{Kn} = \frac{1}{2}\Delta\sum_{k\geq 0}\gamma_n^{(2k)},$$

where we set

$$\begin{aligned}
\beta_n &= \sum_{p=1}^{n-1}(\Gamma_p \wedge \Gamma_{n-p}), \\
\gamma_n^{(0)} &= Gd^*\beta_n^{(0)}, \\
\gamma_n^{(2k+2)} &= Gd^*(\beta_n^{(2k+2)} + C\gamma_n^{(2k)}), \quad k \geq 0.
\end{aligned}$$

We have $\Delta\gamma_n^{(0)} = G\Delta d^*\beta_n^{(0)}$. By Lemma 5.5, $C\gamma^{(2k)} = \mu\Delta\gamma^{(2k)} - \Delta(\mu\gamma^{(2k)})$. Hence for $k \geq 0$,

$$\Delta\gamma_n^{(2k+2)} = G\Delta d^*(\beta_n^{(2k+2)} + \mu\Delta\gamma_n^{(2k)}).$$

Set $\phi_n^{(2k)} = \Delta\gamma_n^{(2k)}$, then

$$\Gamma_{Kn} = \frac{1}{2}\sum_{k\geq 0}\phi_n^{(2k)},$$

where

$$\begin{aligned}
\phi_n^{(0)} &= G\Delta d^*\beta_n^{(0)}, \\
\phi_n^{(2k+2)} &= G\Delta d^*(\beta_n^{(2k+2)} + \mu\phi_n^{(2k)}), \quad k \geq 0.
\end{aligned}$$

The potential function

$$\Phi_K = \int_M \frac{1}{6}\Gamma_K^3 - \frac{1}{4}\Gamma_K^2 \wedge (\Gamma_K - \Gamma_{K1})$$

is a formal power series with coefficients in $S(\mathfrak{k}^*)^K$. There is a ring homomorphism $f : S(\mathfrak{k}^*)^K \to S^0(\mathfrak{k}^*)^K = \mathbb{R}$. Now $f(\Gamma_K)$ and $f(\Phi_K)$ give exactly the formal

Frobenius manifold structure constructed in Cao-Zhou [7]. Hence $\Phi_K$ should be thought of as a family of formal Frobenius manifold structures. This is more clearly seen when $K$ is a torus $T$. There is a deformation family of associative algebraic structures on $H^*(M)$ given by

$$\int_M \frac{1}{6}\Gamma_{T1}^3.$$

$\Phi_T$ then gives a family of solutions to WDVV equations with them as initial data.

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